Milnor strange nonchaotic attractor with complex basin of attraction

H. L. Yang
Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Strasse 38, D-01187 Dresden, Germany
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The transcritical blowout bifurcation of a quasiperiodic torus on an invariant subspace is studied in this paper. We found that the strange nonchaotic attractor (SNA) beyond the blowout bifurcation is only a weak attractor in the sense of Milnor. This is different from the popularly studied case where the Milnor attractor is a chaotic one with a riddled basin of attraction. Characters of this Milnor SNA and the influence of a random noise are studied.

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Recently, systems possessing an invariant subspace (ISS) attracted a great deal of research interest [1–7]. The existence of an ISS requires that a system has some kind of symmetry. A situation where symmetry can appear naturally is the synchronization of coupled identical units [1]. Synchronization phenomena have been extensively studied in the context of laser dynamics, electronic circuits, chemical and biological systems, and secure communication.

For systems possessing an ISS, a central question is the transverse stability of the ISS, i.e., whether an orbit starting from the vicinity of the ISS will finally approach it or escape from it. A quantity measuring this stability is the transverse Lyapunov exponent (TLE), which measures the growth rate of transverse perturbations. If the TLE is negative, a transverse perturbation will decrease gradually and the ISS is transversely stable. Otherwise, it is unstable. The problem turns out to be quite complex when there is a chaotic attractor on the ISS [6] due to the fact that there are infinite number of unstable periodic orbits (UPO's) embedded in it [8]. With a change of some parameter, e.g., the coupling strength for coupled systems, UPO's become transversely unstable at different values of the parameter. The point where the least stable UPO becomes unstable is called a riddling bifurcation [7]. The moment when the whole chaotic attractor becomes transversely unstable is called the blowout bifurcation [5]. In between the riddling and blowout bifurcation, the attractor on the ISS is only a weak attractor in the sense of Milnor [3,9], i.e., its strength is zero while it does attract a set of finite measure in the phase space. This is a typical situation where a Milnor attractor with zero strength is dominant.

In a recent paper [10], Yalçınkaya and Lai studied a situation where in the ISS there is a quasiperiodic torus instead of the chaotic attractor. They showed that the loss of the transverse stability of the torus can lead to the birth of a strange nonchaotic attractor (SNA) [11]. SNA's are attractors which are geometrically strange, i.e., having a fractal structure, while the largest Lyapunov exponent is nonpositive, i.e., having no sensitive dependence on initial conditions. However, no riddled basin of attraction or Milnor attractor was observed prior to the blowout bifurcation in the study of Yalçınkaya and Lai [10]. In this paper, we will go further along this direction to study the *transcritical* blowout bifurcation of a quasiperiodic torus, while the case studied by Yalçınkaya and Lai is a period-doubling one. The motivation for the current work is the following: In previous studies [1,3–7,10], the systems used are of much stronger symmetry than necessary for the existence of an ISS. For example, in the case of synchronization of coupled units, symmetric coupling is usually used [1,7] while the use of identical units is already enough to produce an ISS, independently on the coupling method. In this paper, we address the situation where the unnecessary symmetry is released. To illustrate our finding, we use a simple two-dimensional map which possesses an ISS while the whole system is asymmetric with respect to this ISS. Due to this asymmetry, the quasiperiodic torus in the ISS loses its transverse stability through a transcritical blowout bifurcation. We found that the basin of attraction of the SNA beyond the blowout bifurcation can be penetrated by a dense set of points belonging to the basin of another attractor. The SNA is a different type of weak Milnor attractor in contrast to the case previously reported [3,7]. Characteristics of the basin of attraction of the Milnor SNA and the effect of noise are also studied.

We consider the following general class of dynamical systems popularly used in the literature [7,10],

$$\begin{aligned} x_{n+1} &= f(x_n), \\ y_{n+1} &= F(x_n, p) G(y_n), \end{aligned} \tag{1}$$

where $f(x)$ is a map that has a quasiperiodic torus and p is the control parameter. In previous works [7,10], the function $G(y)$ on the right hand side of the second equation in Eq. (1) is odd, i.e., it has the symmetry $G(-y) = -G(y)$. This symmetry is appropriate for the symmetric coupling in coupled systems [7]. Obviously, this symmetry is not necessary for the presence of the ISS $y=0$. Thus we would like to use a function $G(y)$ which is *neither* even *nor* odd to release this symmetry. For simplicity and easiness of illustration, the following version of Eq. (1) is used:

$$\begin{aligned} x_{n+1} &= x_n + \omega \pmod 1, \\ y_{n+1} &= p|\cos(2\pi x_n)| y_n (1 + a y_n - b y_n^2), \end{aligned} \tag{2}$$

where p, a, and b are positive real constants. For any choice of the control parameters, we have $y_n = 0$ for $n > 1$ if $y_0 = 0$, i.e., $y_n = 0$ is an ISS of the whole system. On this ISS, the system possesses a quasiperiodic torus from the circle map $f(x) = x + \omega \pmod 1$. Throughout this paper, values $a = 0.5$ and $b = 0.3$ are kept fixed, and p is used as a control parameter. Using other values of a and b can lead to different

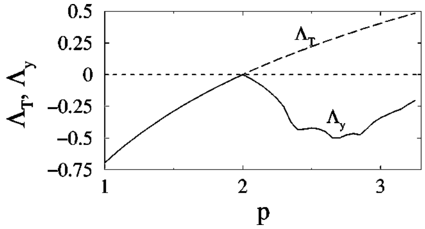

FIG. 1. The variation of Λ_T and Λ_y vs p.

scenarios from the case reported below. Details about these scenarios will be reported elsewhere.

The TLE of the quasiperiodic torus on the ISS is given by $\Lambda_T = \int_0^1 \ln|p\cos(2\pi x)|dx = \ln(p/2)$. The blowout bifurcation occurs as $\Lambda_T = 0$, which defines the critical value for the control parameter $p_c = 2$. It has been argued that beyond the blowout bifurcation [10], there is the possibility that the TLE Λ_T is positive while the largest nontrivial Lyapunov exponent Λ_y of the y subsystem is negative. Results of the numerical calculation for two Lyapunov exponents in our system are shown in Fig. 1. The negative largest nontrivial Lyapunov exponent Λ_y, slightly beyond the blowout bifurcation, warrants that the attractor is nonchaotic. Following the argument in Ref. [11], we can show that it is also geometrically strange: It is obvious that beyond the blowout bifurcation, the ISS $y = 0$ is no longer an attractor, and the new attraction must include some points off the ISS. On the other hand, points $(x = 0.25, y = 0)$ and $(x = 0.75, y = 0)$ must be on the attractor, since the term $\cos(2\pi x)$ is zero for these points. Further forward iterations of these points should also be on the attractor. Therefore, the attractor contains points both *off* and *on* the ISS $y = 0$. Due to the ergodicity, two sets of points are dense in the x direction and interwoven together completely. This, together with the continuity of the attractor, leads to its strangeness.

The attractor and its basin of attraction with $p = 2.3$ is shown in Fig. 2, where points escaping to infinity are denoted by black dots while the basin of the SNA is left blank. Due to the fact that the system is asymmetric with respect to the ISS, the SNA resides only at the negative half plane, and the ISS is a part of its boundary. In Ref. [10], however, the SNA is on both sides of the ISS. It can be seen that the attractor approaches the ISS $y = 0$ from time to time, and has cusp singularities at a dense set of points. This is consistent with our argument that it is a strange attractor. The basin of attraction turns out to be quite complex, and the following characters can be seen from the plot: First, the SNA does attract a set of finite measure in the phase space. The bulk blank region in the negative half plane containing the attractor is part of the basin. Second, a set of points of finite measure belonging to the basin of the attractor at infinity penetrates the ISS, which is a part of the boundary of the SNA. This can be clearly seen from blowing up a part of the basin in the vicinity of the ISS the [upper frame in Fig. 2(a)]. It is expected that a certain perturbation of arbitrarily small strength can kick orbits out of the basin of the SNA, i.e., it is a Milnor attractor with zero strength. This is different from the previously studied case [3] where a chaotic attractor with riddled basin is argued to be a Milnor one. Furthermore, when there is a chaotic attractor on the ISS, the difference in transverse stabilities of UPO's leads to a riddled basin and a Milnor attractor [6]. For the current case, the invariant measure is unique for orbits on the quasiperiodic torus in the ISS. Thus the mechanisms leading to the Milnor attractor should be different. Third, in the basin of attraction of the SNA there are some bands where all points go finally to the SNA (or infinity). This means that the basin of attraction for the SNA is not a riddled one in the original sense of Alexander *et al.* [3]. Fourth, as the ISS is approached, the scale of bands becomes smaller and smaller and eventually goes to zero. In an arbitrarily small region which contains the ISS, there is always a set of points of finite measure going to infinity. In other words, the degree of mixing of the two basins is inhomogeneous, and it becomes higher and higher as the ISS is approached [see Fig. 2(b)]. Finally, all points in the vicinity of the ISS will first escape due to the local instability since the TLE is now positive. Some of them will be mapped to the SNA in the negative half plane due to the global nonlinearity of the system. This is different from the case of a riddled basin where a set of finite measure converges to the ISS monotonically [12].

Effect of noise. Since fluctuations and perturbations are inevitable in real experiments and technical applications, it is in general important to study the influence of noise. In our system, the strength of the Milnor SNA is zero, and the effect of a perturbation will be significant. In principle, perturbations of arbitrarily small strength can kick orbits out of the basin of the Milnor SNA. However, with a decreasing

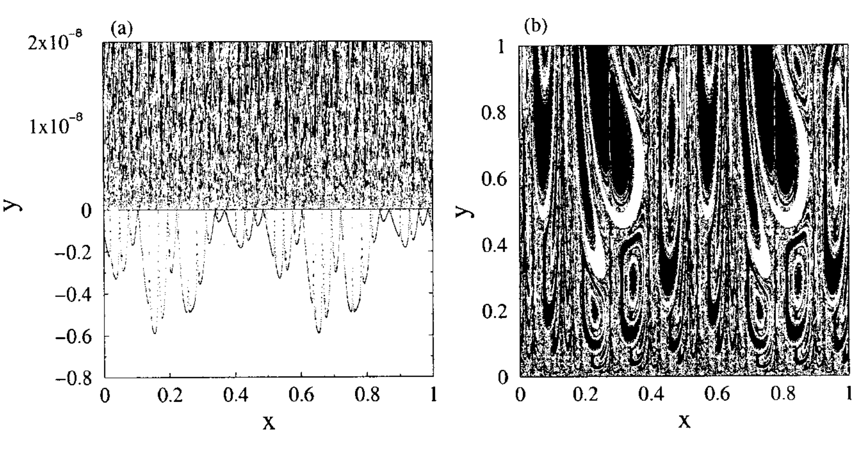

FIG. 2. (a) The SNA and the blowup of part of its basin with $p = 2.3$. (b) The basin at the positive half plane.

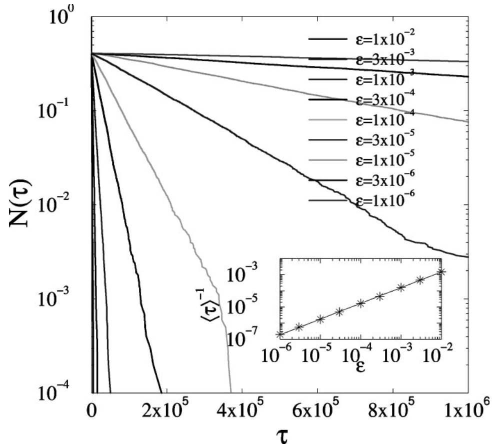

FIG. 3. The normalized number of points $N(\tau)$ left after a transient τ. The inset gives $\langle\tau\rangle^{-1}\sim\epsilon$.

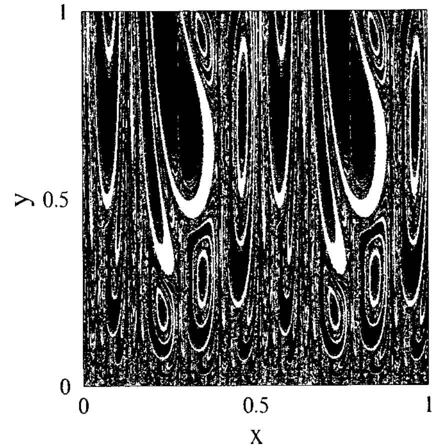

FIG. 5. Points escaping within a transient of $\tau=50$ for $p=2.3$ and $\epsilon=10^{-4}$.

strength of the perturbation, one should wait longer and longer times to see the escaping. To characterize the temporal behavior of this noise-induced escaping, we first calculate the quantity $N(\tau)$, which is the number of orbits left on the attractor after a transient time τ. Usually it has the form $N(\tau)\sim\exp(-\tau/\langle\tau\rangle)$ where $\langle\tau\rangle$ is the mean value of the escape time [13]. The variation of $\langle\tau\rangle$ on ϵ is shown in the inset of Fig. 3. A fit of the data gives $\langle\tau\rangle\sim\epsilon^{-1}$.

We also calculate the probability density function (PDF) $p(\tau)$ for the escape time distribution. From the plot in Fig. 4, it can be seen that $p(\tau)$ decreases exponentially with the increasing of τ. A detail study shows that it is of the form $p(\tau)\sim\epsilon\exp(\epsilon\tau)$. This is consistent with the form of $N(\tau)$ above. Two interesting things need to be pointed out about this PDF $p(\tau)$: First, it has a quickly decreasing part at small τ. The decreasing rate is universal for all cases with different noise strength ϵ. This means that in addition to the characteristic time $\langle\tau\rangle$ there is another time which is not influenced by changing the noise strength. Our explanation for this phenomenon is as follows: In the original system without noise, there are two groups of points. One is in the basin of the Milnor SNA. The other goes eventually to infinity. Under the influence of a very weak noise, *most* points in the last group cannot ''feel'' the noise before they escape to infinity [14]. This contributes to the universal quickly decreasing part at small τ in $p(\tau)$. The first group will initally go to the vicinity of the Milnor SNA, and wander there for a long time before escaping. Since the noise strength ϵ is small, it can only have a significant influence on orbits entering the region $|y|<\epsilon$, say the one in the first group, since the SNA approaches the ISS from time to time. This forms the slowly decreasing noise-dependent part at large τ. To check the fitness of our conjecture, we consider a grid of 3000×3000 points in the regions $0<x<1$ and $0<y<1$. Those escaping within a transient $\tau<50$ with $\epsilon=10^{-4}$ are denoted by black dots in Fig. 5. In comparing with Fig. 2, the similarity is obvious.

The second group is the additional oscillation in $p(\tau)$, besides the main trend is of an exponential decrease. This can be seen in Fig. 4. To make it more clear, we plot the quantity $p(\tau)\exp(\tau/\langle\tau\rangle)$ instead of $p(\tau)$ (see Fig. 6). Spectrum analysis shows that this oscillation is a harmonic of the external quasiperiodic driving. The frequency for the oscillation shown in Fig. 6 is $f\simeq0.236\simeq2\omega$ where $\omega=\sqrt{5}-1/2$ is the frequency of the external driving. Calculations for other cases with different external driving frequencies like

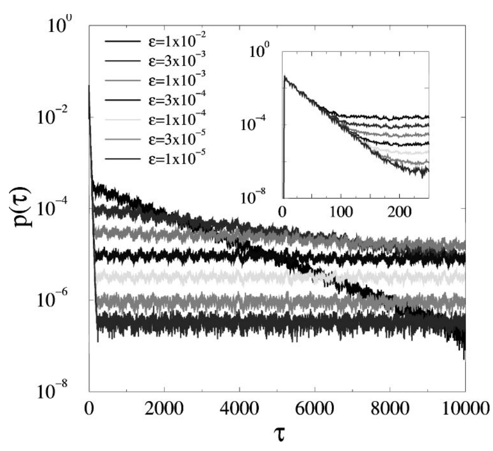

FIG. 4. The PDF $p(\tau)$ of the escape time distribution. The inset shows the part at small τ.

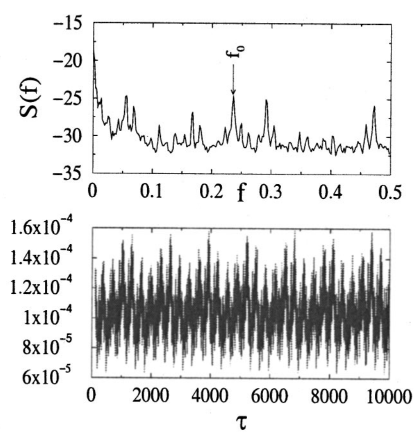

FIG. 6. The variable $p(\tau)\exp(\tau/\langle\tau\rangle)$ vs τ (lower); the spectrum of the signal $p(\tau)\exp(\tau/\langle\tau\rangle)$ (upper).

$\omega = \sqrt{2} - 1$ give similar results. We expect that this additional quasiperiodic oscillating in the PDF of the escape time distribution is a fingerprint of quasiperiodically driven systems.

Finally, we would like to outline the main results of the current work: (1) We relate the SNA and Milnor attractor together in the transcritical blowout bifurcation of a quasiperiodic torus. To our knowledge, this is different from previously reported situations, where the Milnor attractor is a chaotic one with a riddled basin. (2) We find that there are two different time scales for the current system under the influence of noise. This is expected to be common for riddled systems under influence of noise. (3) The additional quasiperiodic oscillation on the PDF of the escape time distribution should be one of the fingerprints of quasiperiodically driven systems.

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